THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH3070 Introduction to Topology 2017-2018
Solution of Tutorial Classwork 3

1. (a) Note that $U = \{x\} \cup \mathbb{R}\backslash\{x_1, x_2, x_3, \dots\}$ is an open set with $x \in U$. By assumption, there exists $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$. Since $x_n \in U$ if and only if $x_n = x$, we have $x_n = x$ for all $n \geq N$.

   (b) Consider the function $f : (\mathbb{R}, \text{cocountable topology}) \to (\mathbb{R}, \text{discrete topology})$ by $f(x) = x$. Suppose $x_n \to x$. Then we have $x_n = x$ for all $n \geq N$. In particular, we have $f(x_n) = f(x)$ for all $n \geq N$. Hence $f(x_n) \to f(x)$ and f is sequentially continuous.

   However, it is not continuous since $f^{-1}(\{0\}) = \{0\}$ is not open under the cocountable topology.

   (c) * Pick any open set $V \in \mathfrak{T}_Y$. Suppose $f^{-1}(V)$ is not open. Then there exists point $x \in f^{-1}(V)$ such that for any open set U with $x \in U$, $U\backslash f^{-1}(V) \neq \emptyset$. Let $\{B_n\}_{n\in\mathbb{N}}$ be a local base at x. Choose $b_n \in (\bigcap_{i=1}^n B_i)\backslash f^{-1}(V)$. One can show that $b_n \to x$ (Try). By sequential continuity, we have $f(b_n) \to f(x)$. In particular, we have $f(b_n) \in V$ when n is sufficiently large. This contradicts with the fact that $b_n \notin f^{-1}(V)$.

2. ($\Rightarrow$) Suppose C is a closed nowhere dense set. Let $U = X\backslash C$. Since C is closed, U is open. Moreover, $\overline{X\backslash U} = \overline{C} = C$ and $\overline{U} = X\backslash \mathring{C} = X\backslash \mathring{\overline{C}} = X\backslash\emptyset = X$. Hence $C = \overline{U} \cap \overline{X\backslash U}$.

   ($\Leftarrow$) Suppose $C = \overline{U} \cap \overline{X\backslash U}$ for some open set U. Then since U is open, $X\backslash U$ is closed and we have $C = \overline{U} \cap X\backslash U$. Hence $\mathring{\overline{C}} = \mathring{C} = (\overline{U} \cap X\backslash U)^{\circ} = \mathring{\overline{U}} \cap (X\backslash U)^{\circ} = \mathring{\overline{U}} \cap X\backslash\overline{U} = \emptyset$.

3. ($\Rightarrow$) Given that X is of second category. Suppose we have a countable collection of open dense set $\{D_k\}_{k\in\mathbb{N}}$. Then $\{X\backslash D_k\}_{k\in\mathbb{N}}$ is a countable collection of closed nowhere dense set. Since X is of second category, by definition we have $X \neq \bigcup_{k\in\mathbb{N}} X\backslash D_k$. Since $\bigcup_{k\in\mathbb{N}} X\backslash D_k = X\backslash \bigcap_{k\in\mathbb{N}} D_k$, we have $\bigcap_{k\in\mathbb{N}} D_k \neq \emptyset$.

   ($\Leftarrow$) Let $\{U_k\}_{k\in\mathbb{N}}$ be a countable collection of nowhere dense set. Then $\{\overline{U_k}\}_{k\in\mathbb{N}}$ is a countable collection of closed nowhere dense set. Hence $\{X\backslash\overline{U_k}\}_{k\in\mathbb{N}}$ is a collection of open dense set. By assumption, we have $X\backslash \bigcup_{k\in\mathbb{N}} \overline{U_k} = \bigcap_{k\in\mathbb{N}} X\backslash\overline{U_k} \neq \emptyset$. Hence $X \neq \bigcup_{k\in\mathbb{N}} \overline{U_k}$ and X is of second category.